AIAA 79-1740R

Optimal Local Control of Flexible Structures

David B. Schaechter*
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif.

Local control is defined as a control law that includes feedback of only those state variables that are physically near a particular actuator. The necessary condition for the solution of the linear quadratic optimal control problem with the constraint of local state feedback is derived. This development results in a systematic approach for reducing the complexity of the control law for high order systems. The local control techniques are extended to include dynamic systems obtained directly from the finite-element modeling approach. Numerical examples of the local control of 1) a simple two mass system and 2) a free-free flexible beam are presented.

Nomenclature

A	= state weighting matrix
B	= control weighting matrix
C	= control gain matrix
EI	= flexural rigidity
F	= open-loop dynamics matrix
G	= control distribution matrix
$\mathcal{H}$	= Hamiltonian
I, I_n	= $n \times n$ identity matrix
J	= performance index
K	= stiffness matrix
L	= length
M	= mass matrix
q	= modal amplitude vector
Q	= spectral density of w
S	= sweep solution matrix
t, t_0, t_f	= time, initial time, final time
u	= control vector
w	= Gaussian random process vector
x	= state vector
X	= state covariance matrix
λ	= Lagrange multiplier vector
Λ	= Lagrange multiplier matrix
μ	= Lagrange multiplier matrix control constraint
Ω	= diagonal matrix of natural frequencies
ϕ	= eigenvector matrix
ρ	= density
$(\ )^T$	= transpose
$(\dot{\ })$	= time derivative
E( )	= expected value
$(\ )^{-1}$	= matrix inverse

Introduction

As space structures increase in size, the complexity of the vehicle dynamics, as measured by the number of states that are required to model these vehicles, increases. Size alone is not the only driver of model complexity, however. As more demands are made of closed-loop system performance, vehicles which heretofore have been modeled as "rigid" bodies must now be modeled as flexible bodies. Unfortunately, the capability of implementing complex on-board controllers has not increased at the same pace. The problem presented to the control system engineer is to develop efficient techniques for designing simple control laws, given a complex dynamic system.

This paper contains a derivation of the necessary conditions for an optimal local control law for a general system in state variable format. A local control law is one in which only local state information is used to synthesize the control law for each actuator (even if additional state information is available) and, hence, is a different problem from that of output feedback.[1,2] A practical application of this idea occurs in designing control systems for large flexible space structures, where many sensor outputs may be available for feedback, yet it is not practical to do so due to the large spatial distances involved.

By representing the structure in physical coordinates (the initial output of the finite-element analysis) rather than modal coordinates, the numerical solution of the optimal full state and local feedback problems may be simplified. Numerical examples of control law designs for a simple two mass model and for a free-free flexible beam are given. Similar examples using different design approaches may be found in Refs. 3 and 4. By retaining physical coordinates, it is shown that the optimal local control law for the free-free flexible beam is equivalent to the addition of "active structural stiffness" and "active structural damping," a result which may prove useful for designing control systems for general large structures.

The counterpart to the local control law is a local estimation scheme. Local estimation processes sensor information using a dynamic model of the system to obtain estimates of only the nearby components of the state vector, not the entire state vector. These components of the estimated state vector should be precisely those components required by the local control law. Combining the local state estimator with the local control law results in a local controller which significantly reduces the amount of on-board computation, and allows the computations to be performed in a distributed or parallel manner. Only the control law design portion of the local controller will be considered in this work.

Derivation of the Optimal Local Control

The derivation of the necessary conditions for the optimal local control begins with a system in state variable format and a quadratic performance index to be minimized as is shown in Eq. (1):

$$\dot{x} = Fx + Gu + w \qquad x(t_0) \quad \text{given}$$

$$\min_u J = \frac{1}{2}\int_{t_0}^{t_f} (x^T A x + u^T B u)\,dt \tag{1}$$

Presented as Paper 79-1740 at the AIAA Guidance and Control Conference, Boulder, Colo., Aug. 6-8, 1979; submitted Oct. 19, 1979; revision received April 21, 1979. This paper is declared a work of the U.S. Government and therefore is in the public domain.

*Member of Technical Staff. Member AIAA.

Assuming that the desired final solution is of the form $u=Cx$, the substitution into Eq. (1) is made yielding the deterministic equivalent

$$\dot{x}=(F+GC)x \qquad x(t_0) \text{ given}$$

$$\min_C J=\frac{1}{2}\int_{t_0}^{t_f} x^T(A+C^TBC)x\mathrm{d}t \tag{2}$$

At this point, the constraint that only local states be fed back can be enforced by requiring that certain components of C be identically zero and that the minimization in Eq. (2) be carried out with respect to the remaining, nonzero components of C. Equivalently, the constraints may be adjoined to the Hamiltonian to yield

$$\mathcal{H}=\frac{1}{2}x^T(A+C^TBC)x+\lambda^T(F+GC)x+\mu_{ij}C_{ij} \tag{3}$$

where

$$\mu_{ij}=0 \quad \text{if} \quad C_{ij}\neq 0$$

$$\mu_{ij}\neq 0 \quad \text{if} \quad C_{ij}=0$$

and where summation over the repeated indices is implied. The optimality condition becomes

$$BCxx^T+G^T\lambda x^T+\mu=0 \tag{4}$$

where the μ_{ij}'s are picked to make the corresponding constrained C_{ij}'s equal to zero.

The usual sweep solution[5] obtained by letting

$$\lambda(t)=S(t)x(t) \qquad C=-B^{-1}G^TS(t)$$

no longer works in general since the optimality condition in Eq. (4) results in C being a function of x and t, i.e., the minimization can no longer be performed independently of the initial condition. However, the appearance of the terms xx^T and λx^T above suggests that a linear statistically optimal control may exist. The stochastic analog of Eq. (2) in terms of the covariance of the state is[5]

$$\dot{X}=(F+GC)X+X(F+GC)^T+Q \qquad X(t_0) \text{ given}$$

$$\min_C J=\text{trace}\ \frac{1}{2}\int_{t_0}^{t_f}(AX+C^TBCX)\mathrm{d}t \tag{5}$$

where designated C_{ij}'s are zero and $X=\mathrm{E}(xx^T)$. The accomplishment in the preceding step is to average the performance index over a range of possible initial conditions. Rather than considering all initial conditions to be equally likely,[6] a more realistic range of possible initial states can be obtained using the state covariance matrix. The adjoint matrix equation and optimality condition are

$$-\dot{\Lambda}=\Lambda(F+GC)+(F+GC)^T\Lambda+A+C^TBC \quad \Lambda(t_f)=0$$

$$BCX+G^T\Lambda X+\mu=0 \tag{6}$$

Although an exact solution to Eq. (6) is possible, an approximate solution may be obtained easily by expanding the equations in Eq. (6) to first order in μ about the optimal solution for $\mu=0$. This yields

$$C(t)\cong -B^{-1}[G^T\Lambda(t)+\mu X^{-1}(t)] \tag{7}$$

where Λ and X are the solutions of the unconstrained optimal control problem ($\mu=0$), and μ is picked to zero the corresponding components of C. For the unconstrained problem, $\mu=0$, the result in Eq. (7) reduces to the familiar result, $C=-B^{-1}G^T\Lambda$. Equation (7) has a nice physical interpretation. To first order in μ, if not all the states can be fed back, those states that are available should be fed back with a correction to the feedback gains based on the correlation between those states fed back, and the remaining states. (As a practical note, the inverse in Eq. (7) need never be computed. In fact only a few elements of X need to be manipulated.) Furthermore, since the solution is expanded about the optimal solution, it can be shown that $\partial J/\partial\mu_{ij}=0$, i.e, this concept of local control does not severely affect performance to first order in μ. The algorithm for solving the local control problem can be outlined as follows. First, solve the full state optimal control problem, next, apply the local control correction appearing in Eq. (7).

It should be noted that as in the full state feedback case, for F, G, A, B, and Q all constant, it is possible that a steady-state solution for C, Λ, and X may be obtained as $t_f-t_0\rightarrow\infty$. However, as opposed to the case of full state feedback, stability of the closed-loop system is not guaranteed when using local control gains. The eigenvalues of $F+GC$ using the local control gains must be determined to verify stability of the closed-loop system.

Thus far, the problem of solving the full state optimal control problem (step one) for high-order systems has been avoided. However, the solution procedure required usually assumes that a low-order model of the high-order system is available to make the problem tractable. Traditionally, the structural analyst supplies as many modes as the control system designer wishes. Regardless of where the actual truncation occurs, or by what method, the control system designer begins with a deficient model. This may result in closed-loop instabilities.[7]

The alternative to working with the truncated system model is working with a full order finite-element model, as it is generated by the structural analyst. The next section offers some hope that the analysis of these high-order systems, particularly structural systems, may be feasible.

Finite-Element Structural Models

In an attempt to develop control system design techniques for high-order systems, and to alleviate the problem of truncated modes, it is worthwhile to examine the structural equations of motion. Attitude control, stationkeeping, and figure control of many structures can be represented by the following matrix equation:

$$M\ddot{x}+Kx=Gu \tag{8}$$

Matrix bandedness of M and K is a direct result of the finite-element modeling. Since control inputs from a given actuator are applied at a single station on the structure, nearly all of the elements of G are zero.

Equation (8) can be placed in modal form

$$\ddot{q}+\Omega^2q=\phi^TGu \tag{9}$$

by normalizing the eigenvector matrix ϕ so that

$$x=\phi q \qquad \phi^TM\phi=I \qquad \phi^TK\phi=\Omega^2$$

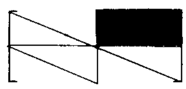
where Ω is a diagonal matrix of modal frequencies. Furthermore, computer programs like EIGSOL[8] and DAMP[9] can solve the open-loop eigenvalue problem [$u=0$ in Eq. (8)] very efficiently by taking full advantage of the matrix sparsity in both storage and computation.

Consider now the problem of designing a control system for the system in Eq. (9). Selection of a control law can be based on the minimization of a quadratic performance index similar to that appearing in Eq. (1).

$$J=\frac{1}{2}\int_{t_0}^{t_f}(q^TAq+u^TBu)\mathrm{d}2t \tag{10}$$

Regardless of where or if truncation of the modal system occurs, the open-loop system dynamics matrix Ω^2 is a diagonal matrix and, hence, simple to manipulate from a computational point of view. However, the corresponding control distribution matrix $\phi^T G$ is not a sparse matrix, and so the Hamiltonian system for the corresponding optimal control problems has the following form:

$$\begin{bmatrix} \Omega^2 & \phi^T G B^{-1} G^T \phi \\ A & \Omega^2 \end{bmatrix} \longleftrightarrow \tag{11}$$



The shaded areas in Eq. (11) represent nonzero matrix entries. Because little useful matrix structure remains in Eq. (11), the eigensystem analysis required for the solution of the optimal control problem can not be performed efficiently. Consider, instead, the control of the original dynamic system in Eq. (8). At this point the concept of local control emerges naturally. The dynamics of the flexible structure are characterized locally. This is the reason that a good dynamic model of the flexible structure can be obtained using tightly banded matrices. Furthermore, actuators produce effects locally, and sensors measure local behavior. It therefore seems plausible that a good controller may be possible using only local state information.

The optimal control problem may be formulated as follows:

$$M\ddot{x}+Kx=Gu \qquad x(t_0),\dot{x}(t_0) \quad \text{given}$$

$$\min_u J=\frac{1}{2}\int_{t_0}^{t_f}(x^T Ax+u^T Bu)\,dt \tag{12}$$

By adjoining the constraints in Eq. (12) to the performance index with Lagrange multipliers λ and integrating by parts, two times, the closed-loop system dynamics and the corresponding matrix structure are given by

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\lambda} \end{bmatrix}+\begin{bmatrix} K & -GB^{-1}G^T \\ A & K \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x(t_0),\dot{x}(t_0) \quad \text{given} \qquad \lambda(t_f)=\dot{\lambda}(t_f)=0 \tag{13}$$

Equation (13) is the Hamiltonian system corresponding to the optimal control problem in Eq. (12). The important feature to notice is that nearly all of the original system's matrix structure is preserved, and that the eigensystem analysis that must be performed in Eq. (13) to obtain the optimal control differs from that which must be performed in Eq. (12) to analyze the open-loop system dynamics, by only two "stripes" off the principal diagonal. As such, from a computational point of view, both storage and computation time can be reduced by fully exploiting the high degree of matrix sparsity in an eigensystem analysis or through efficient matrix perturbation techniques.

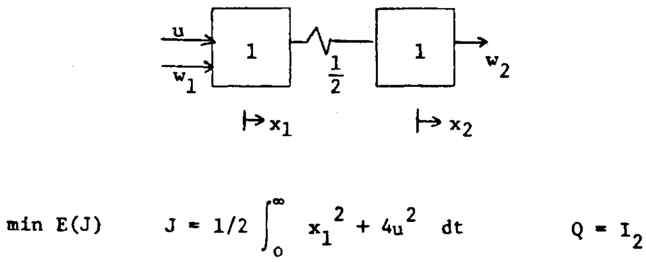


Fig. 1 Two mass model system.

Examples of Local Control Systems

This section contains two examples of the steady-state optimal local control concept and comparisons of the performance of the optimal local control law with the performance of control laws designed using various other approaches.

Example 1: Two Mass Model

Example 1 consists of two unit masses connected by a spring with stiffness, one half (see Fig. 1). The open-loop system has a single rigid body translation mode and a single vibration mode. It is desired to control the position of mass 1 (x_1) with a control input u, in the presence of the disturbances w.

Three approaches for designing control systems are examined: 1) full state optimal control—all four states are available for feedback (x_1, $\dot{x}_1$, x_2, $\dot{x}_2$); 2) modal control—the control system is designed on the basis of a rigid body model; and 3) local control—the feedback gains on x_2 and $\dot{x}_2$ are constrained to be zero.

For the given performance index, the results are most easily summarized in Fig. 2 and Table 1.

The closed-loop eigenvalues, feedback gains, and performance index are given for each control system design approach. Of course, the full state optimal control law performs the best. The desired root locations for the control system designed using the truncated dynamic model are shown in Fig. 2 with triangles. Due to the presence of the truncated system dynamics (in this case the vibration mode), the four closed-loop roots actually end up some distance away from the rigid-body design point. The local control system is designed using the full system dynamics, but with the constraint of partial state feedback. This closed-loop system is "closer" to the optimal full state control law than is the rigid-body control law in terms of both the system performance and

Table 1 Control system design results

	Eigenvalues		Gains				Performance
Controller	Mode 1	Mode 2	x_1	$\dot{x}_1$	x_2	$\dot{x}_2$	index J
Open loop	0,0	$\pm 1j$	0	0	0	0	∞
Full state	$-0.310\pm 0.404j$	$-0.128\pm 0.974j$	−0.384	−0.876	−0.116	−0.453	3.792
Rigid body	$-0.357\pm 0.619j$	$-0.355\pm 0.606j$	−0.500	−1.414	0	0	4.949
Local	$-0.144\pm 0.299j$	$-0.186\pm 0.984j$	−0.221	−0.661	0	0	4.393

the final root locations, and only uses feedback of x_1 and $\dot{x}_1$. It should be noted that as the cost of control decreases (B decreases), the relative merits of the local control law over the rigid-body control law become even more apparent.

Example 2: Free-Free Flexible Beam

The partial differential equation of motion for a free-free flexible beam with constant properties per unit length is given in Fig. 3.

A discretization of this beam can be obtained by choosing a state vector composed of deflections and deflection rates at ten stations along the beam (see Fig. 4). Furthermore, it is assumed that control forces u_i can be applied at the designated stations. Penalizing the beam displacements at each of the ten stations results in a control law which performs stationkeeping, attitude control, and shape control. The matrices $A = I_{10}$ and $B = 0.01 \cdot I_4$ were chosen for the quadratic performance index; as before, $Q = I_{10}$. A full state optimal control law and a local control law were designed for the flexible beam using $\rho = EI = 1$, $L = 9$. Due to the symmetrical placement of the actuators, it is sufficient to present the feedback gains for synthesizing u_1 and u_2. In each of the accompanying figures (Figs. 5-8) solid lines are the full state feedback gains as a function of station location and the broken lines are the local feedback gains obtained under the constraint that only states which are immediately adjacent to the actuator are allowed to be fed back.

There are several important features of these results to recognize.

1) The full state feedback control law makes very little use of distant state information. This feature is not apparent from the modal approach where the feedback gains corresponding to the various modes may be roughly the same magnitude. Evidently the modal feedback effects tend to accumulate at the actuators and cancel far from the actuator.

2) The local control law is equivalent to active springs and dashpots, to provide stationkeeping and attitude control, plus active structural stiffness and active structural damping (i.e., forces proportional to the beam curvature and curvature rate) for shape control. This result may be useful for control system design for general flexible structures.

3) In this example, the difference between the performance index using the local control in place of the full state optimal control is less than 1%. In general, the use of local control guarantees no first-order change in the performance index.

4) From a computational point of view, the following computer CPU times were required by a UNIVAC 1108 to obtain the control laws for this 20-state system: a) OPTSYS[10] = 18 s—the QR algorithm is applied to the 40th-order Hamiltonian system to extract closed-loop eigenvalues

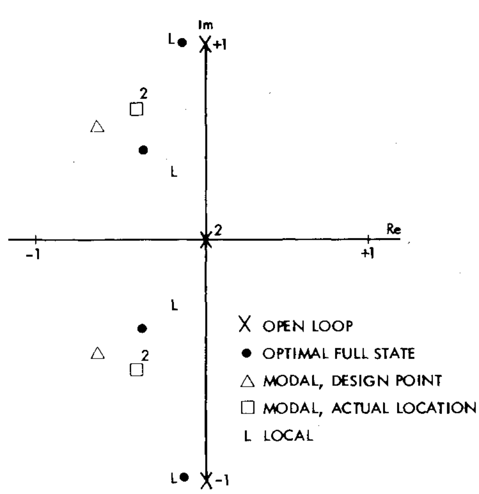
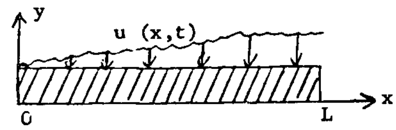


Fig. 2 Frequency domain root locations.



$$\rho \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = U(x,t) \qquad \frac{\partial^2 y}{\partial x^2}(0) = \frac{\partial^2 y}{\partial x^2}(L) = 0$$

$$\frac{\partial^3 y}{\partial x^3}(0) = \frac{\partial^3 y}{\partial x^3}(L) = 0$$

Fig. 3 Free-free flexible beam.

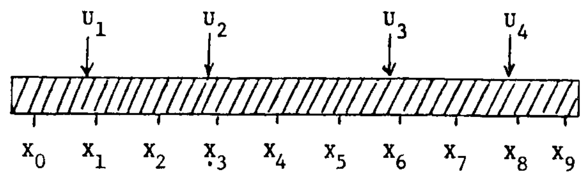


Fig. 4 Discretized beam model.

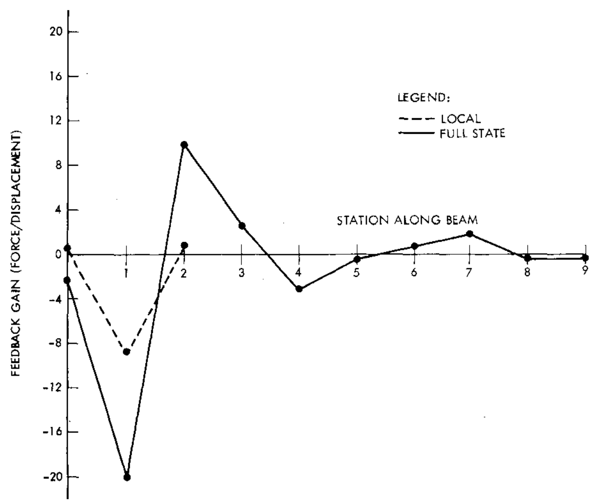


Fig. 5 Full state and local position feedback to actuator 1.

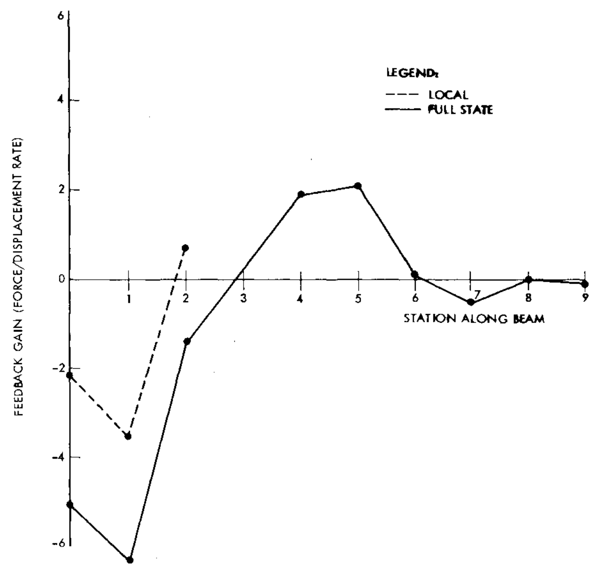


Fig. 6 Full state and local position rate feedback to actuator 1.

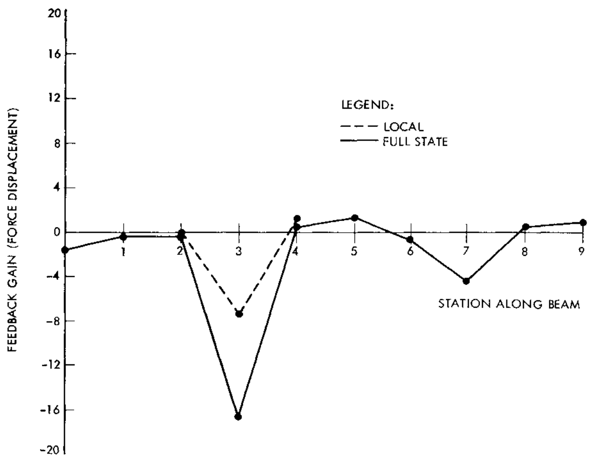


Fig. 7 Full state and local position feedback to actuator 2.

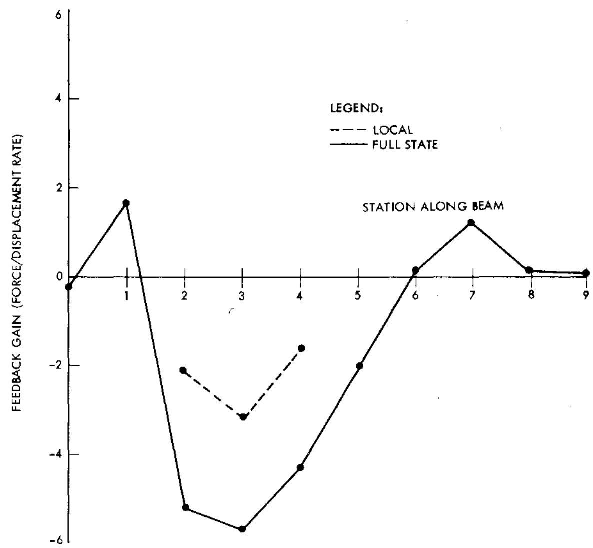


Fig. 8 Full state and local position rate feedback to actuator 2.

and eigenvectors; and b) direct integration = 15 s—direct integration of the matrix Ricatti equation exploiting all matrix sparsity was employed to obtain steady-state gains.

Conclusions

The necessary conditions for the optimal local control law have been derived. Two examples have shown that local control can reduce the complexity of the control law with only a very small sacrifice in the closed-loop performance as compared with a full state feedback control law. In contrast to this, the two-mass example employing local control resulted in improved closed-loop performance as compared with the full state control law based on truncated system dynamics. The results of applying the local control concept to a flexible beam indicate that forces proportional to the beam curvature and the beam curvature rate are synthesized from the local state information for shape control. A direct integration approach to solving for the local control gains has been found to be more efficient than the general eigensystem analysis of the corresponding Hamiltonian system. This improvement is a direct result of fully exploiting the matrix sparsity of the local control formulation. Sparse matrix eigensystem routines and matrix perturbation techniques are currently being studied as a way of further improving the numerical solution of the local control law for systems modeled with finite elements.

Acknowledgments

The research in this paper was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under NASA Contract NAS7-100.

References

[1] Knox, J.R. and McCarty, J.M., "Algorithms for Computation of Optimal Constrained Output Feedback for Multivariable Flight Control System," AIAA Guidance and Control Conference, Palo Alto, Calif., Aug. 1978.

[2] Kosut, R., "Suboptimal Control of Linear Time-Invariant Systems Subject to Control Structure Constraints," *IEEE Transactions on Automatic Control-15,* 1970, pp. 557-563.

[3] Spalding, G.R., "A State-Space Model for Distributed System," AIAA Astrodynamics Conference, Palo Alto, Calif., Aug. 1978.

[4] Ginter, S., and Balas, M., "Attitude Stabilization of Large Flexible Spacecraft," AIAA Guidance and Control Conference, Palo Alto, Calif., Aug. 1978.

[5] Bryson, A.E., and Ho., Y.C., *Applied Optimal Control,* Blaisdell, Waltham, Mass., 1969.

[6] Levine, N., and Athans, M., "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems," *IEEE Transactions on Automatic Control-15,* 1970, pp. 44-448.

[7] Balas, M.J., "Active Control of Flexible Systems," AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Va., June 1977.

[8] Gupta, K.K., "Development of the Program EIGSOL – A Generalized Eigenproblem Solution Routine," Jet Propulsion Laboratory, Pasadena, Calif., Interoffice Memo 3545:78:310, Nov. 1978.

[9] Gupta, K.K., "Eigenproblem Solution of Damped Structural Systems," *International Journal of Numerical Methods in Engineering,* Vol. 8, 1974, pp. 887-911.

[10] Bryson, A.E., and Hall, W.E. Jr., "Optimal Control and Filter Synthesis by Eigenvector Decomposition," SUDAAR 436, Stanford Univ., Dec. 1971.